NETWORK EQUILIBRIUM UNDER CUMULATIVE PROSPECT THEORY WITH ENDOGENOUS STOCHASTIC DEMAND & SUPPLY

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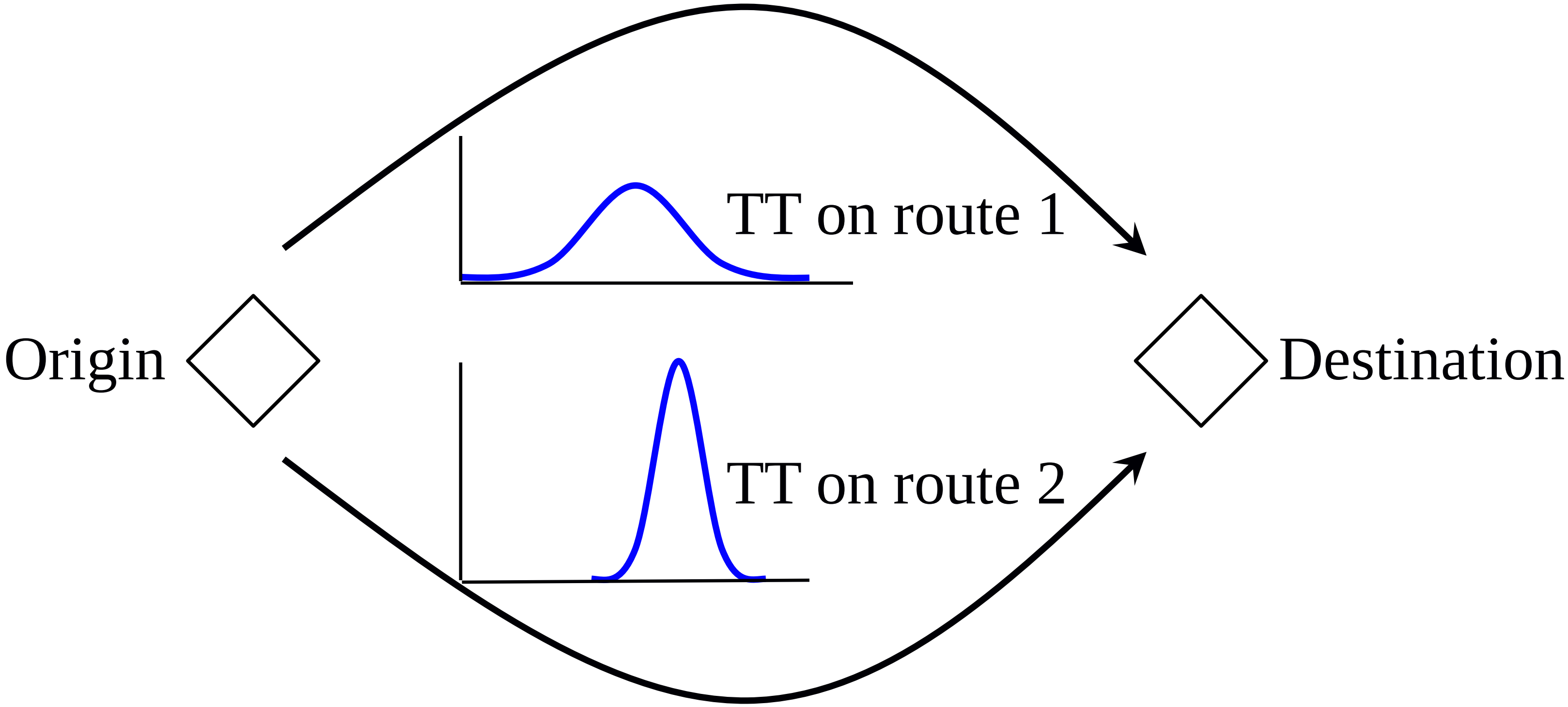
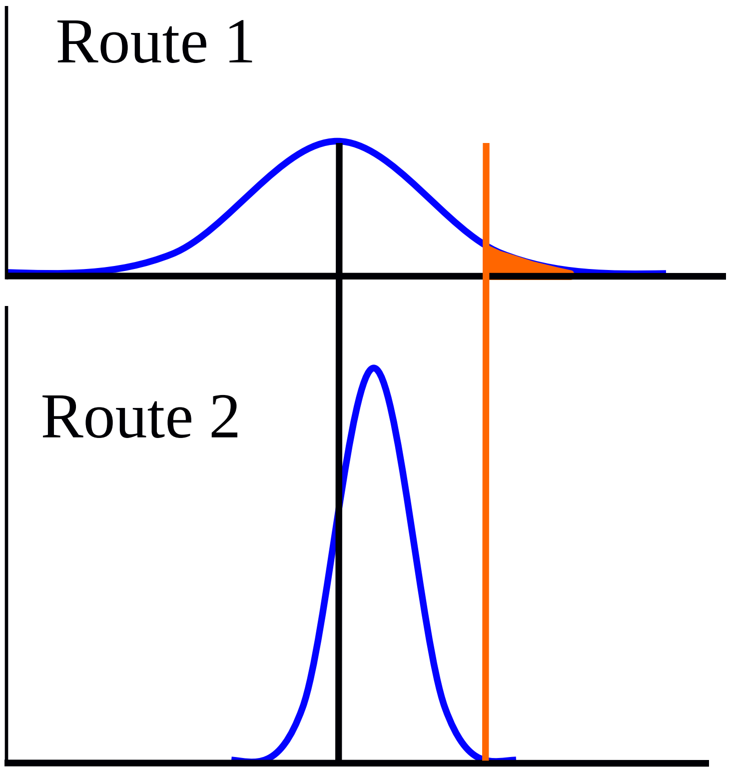
Based on the paper Sumalee et al (2009) presented at ISTTT

Two-link Network

How do drivers choose between routes with uncertain travel times?



Approaches



Mean Travel Time ⇒ Route 1

Prob Late Arrival ⇒ Route 2

Mean + stdDev

Mean + Variance

95th Percentile of TT distribution

…

Empirical Evidence leads to PT

- Choice modellers have shown that *decisions under uncertainty* do not conform to maximising the expected benefit.
- The empirical evidence suggests two major modifications to ‘Expected Utility Maximisation’:
  - the carriers of value are gains/losses relative to a reference point
  - the value of each outcome is multiplied by a decision weight, not by an additive probability.

This generalisation of EUmax is Prospect Theory

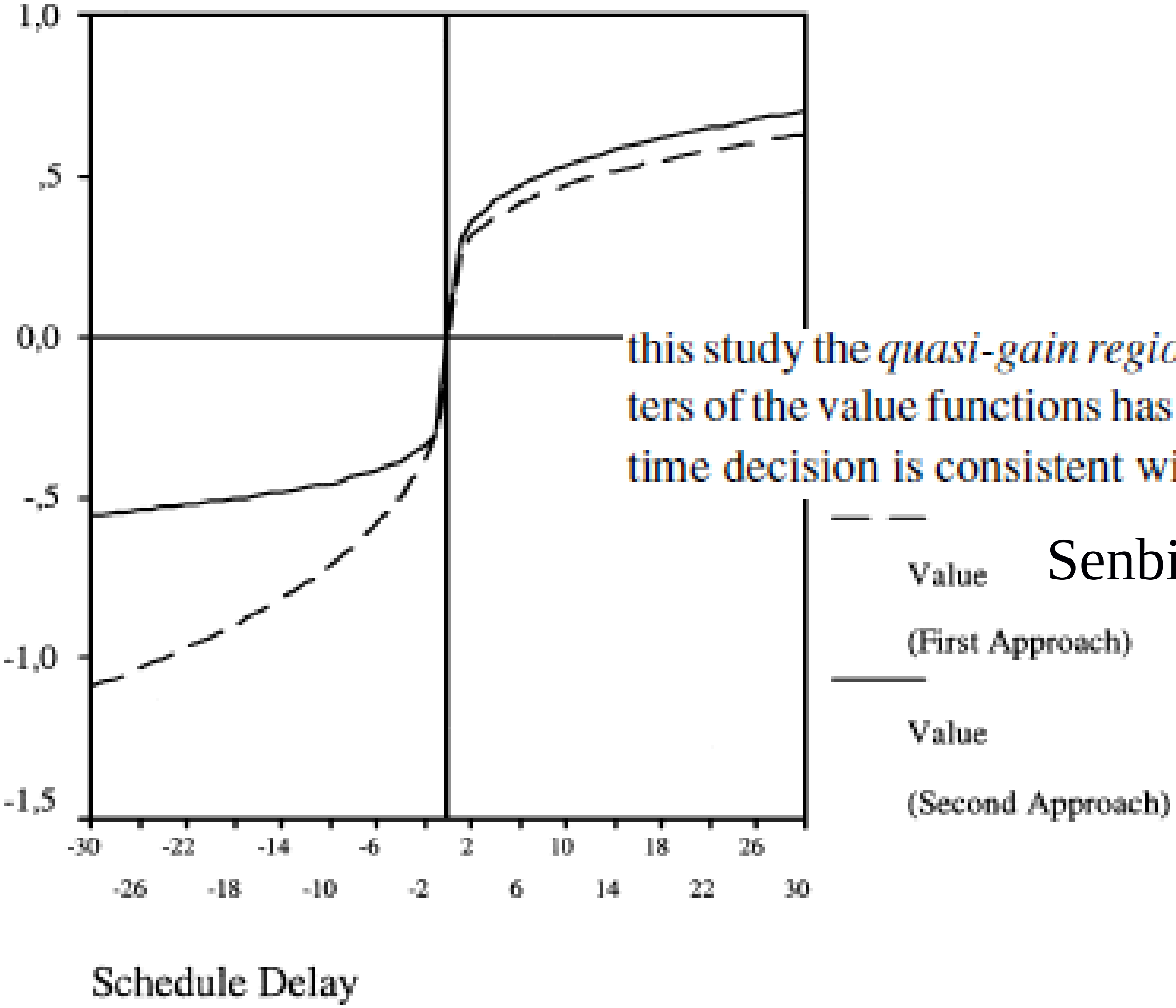
Context

- A traffic network has many sources of variability in demand and supply: results in **T**ravel **T**ime **V**ariability.
- Drivers are aware that travel times are *uncertain* and include this TTV in their decision making.

- Senbil, M., & **Kitamura, R.**, (2004). Reference points in commuter departure time choice: a prospect theoretic test of alternative decision frames. *Journal of Intelligent Transportation Systems 8*, 19–31
- Jou, R.C., **Kitamura, R.**, Weng, M.C., Chen, C.C., (2008). Dynamic commuter departure time choice under uncertainty. Transportation Research Part A 42(5), 774-783.

assumed to reduce the likelihood of choosing that departure time. The empirical results indicate that around 20% of commuters are likely to switch their departure times and routes and most of commuters experience gains, and that preferred arrival times of commuters tend to be near their work starting times. Most importantly, it is shown that, consistent with prospect theory, commuters react asymmetrically to gains and losses.

Empirical evidence

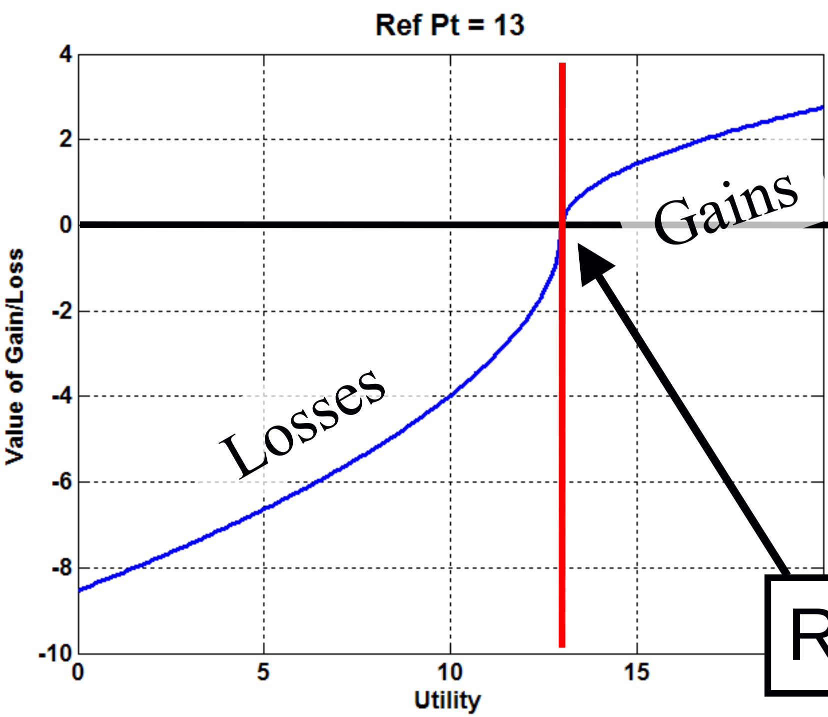


this study the *quasi-gain region*. Estimation of the parameters of the value functions has also indicated that departure time decision is consistent with prospect theory.

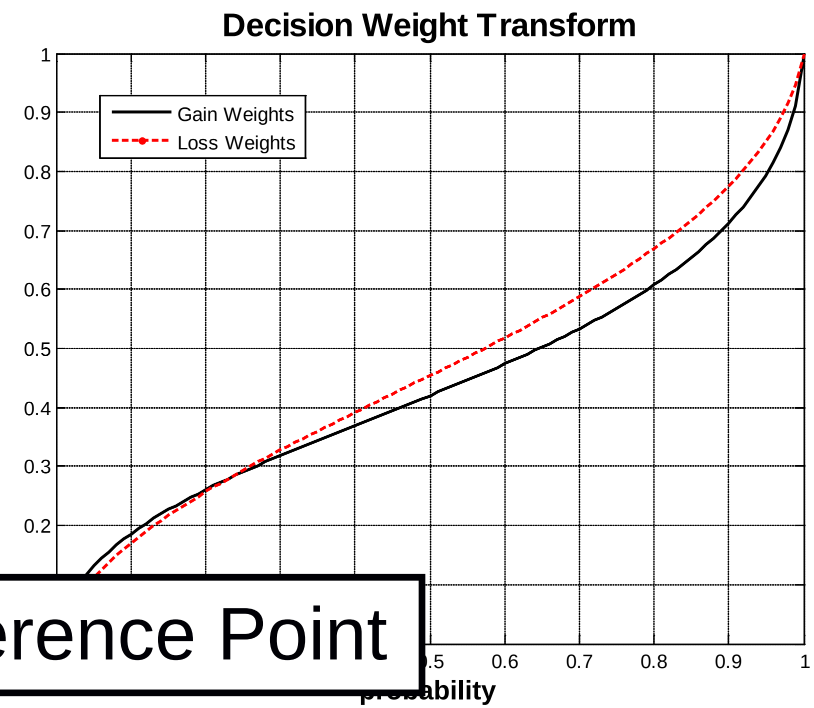
Senbil and Kitamura (2004)

The fundamental embodiment (and impact) of PT is through the following two transformations:

Value Function *g*(.)



Probability Weighting Function *w*(.)



Reference Dependence

Empirical studies show that people's preferences over final outcomes depend on the reference point from which they are judged:

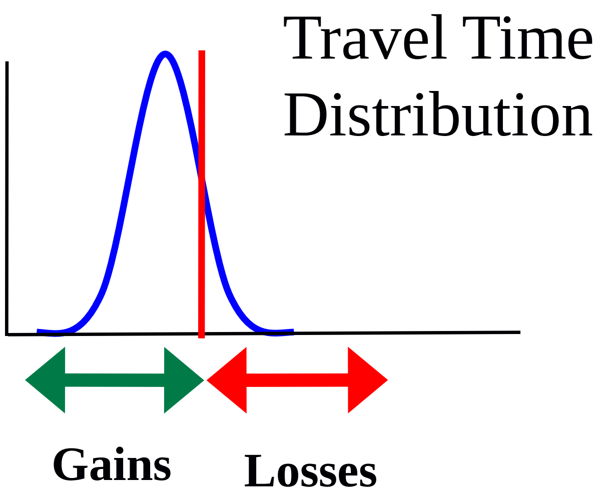
Kahneman & Tversky (1979)	for decision under risk
Kahneman *et al* (1990)	for choice among commodity bundles
Loewenstein & Prelec (1992)	for inter-temporal choice
Bateman *et al* (1997)	for contingent valuation
Dolan & Robinson (2001)	for welfare theory
Bleichrodt & Pinto (2002)	for multi-attribute utility
Senbil&Kitamura (2004)	for departure time choice
Jou, Kitamura et al (2008)	for departure time choice

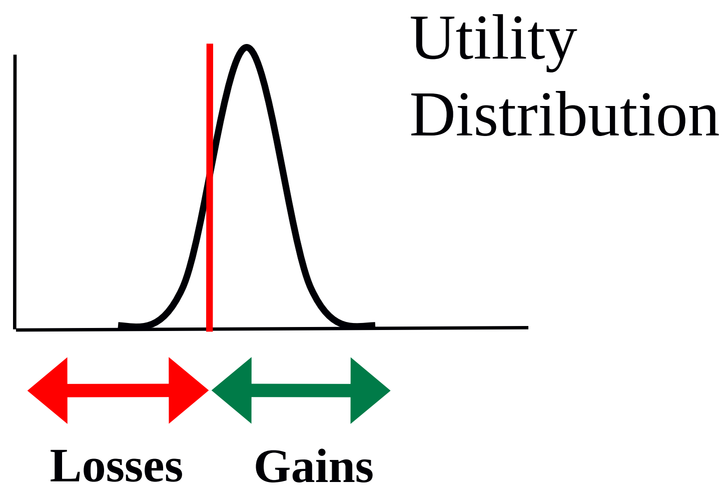
Travel Time → Utility

If we have a distribution of path costs $C_k \sim N\left(c_k\left(\mathbf{x}\right), \sigma_k\right)$

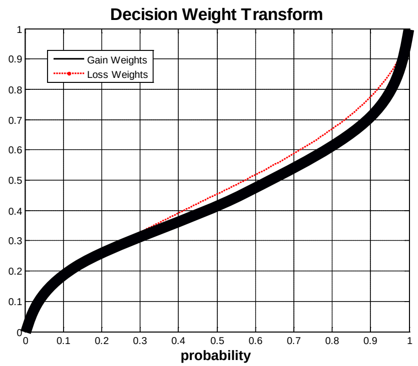
It is convenient to consider the utility

$$U_k = U_{dest} - C_k = U_{dest} - c_k\left(\mathbf{x}\right) - \varepsilon_k$$

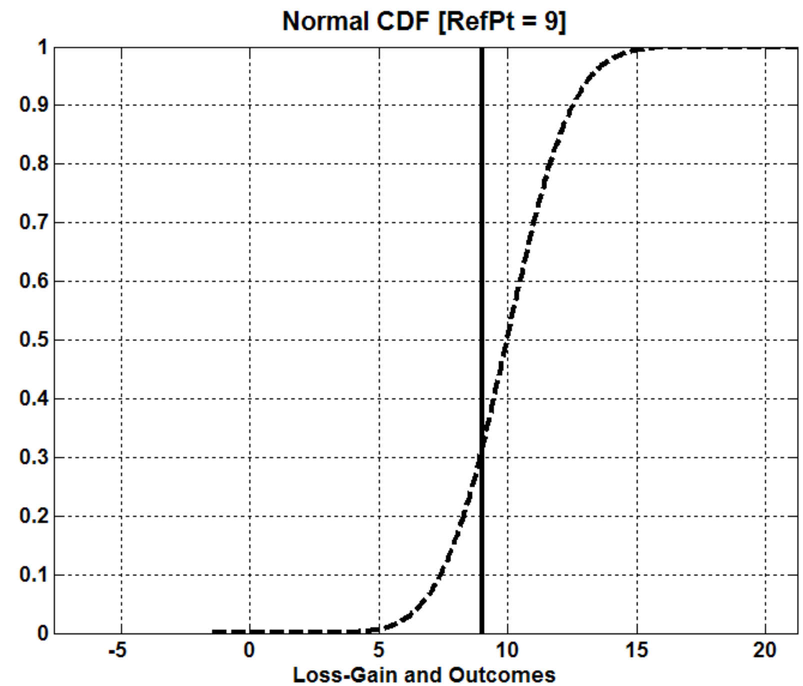




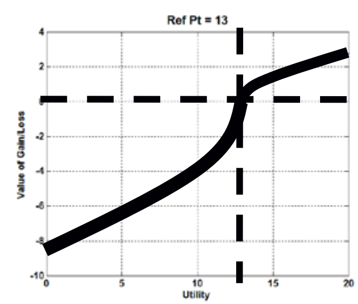
Applying the CPT Transforms



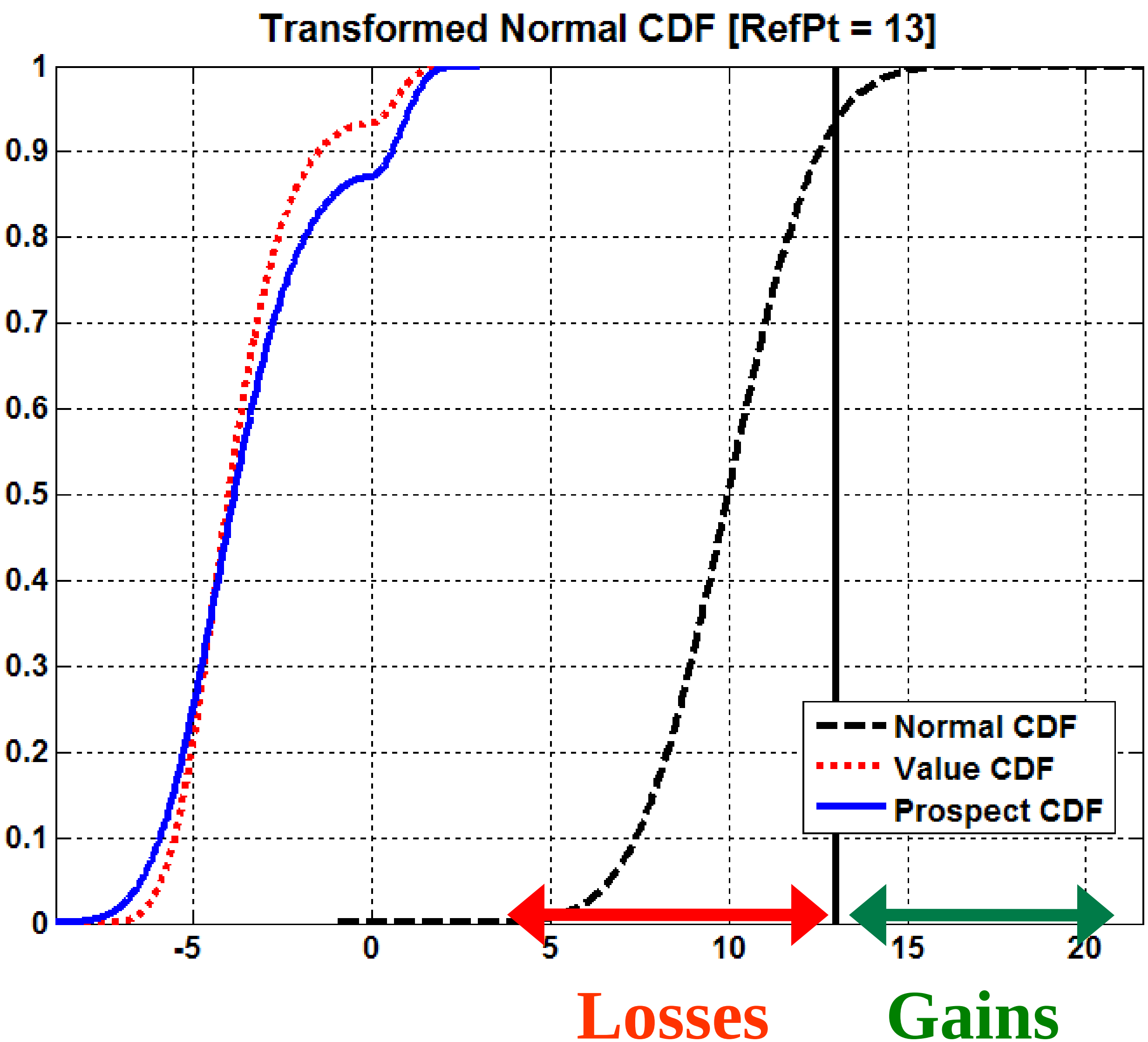
Weighting Function Maps Probabilities



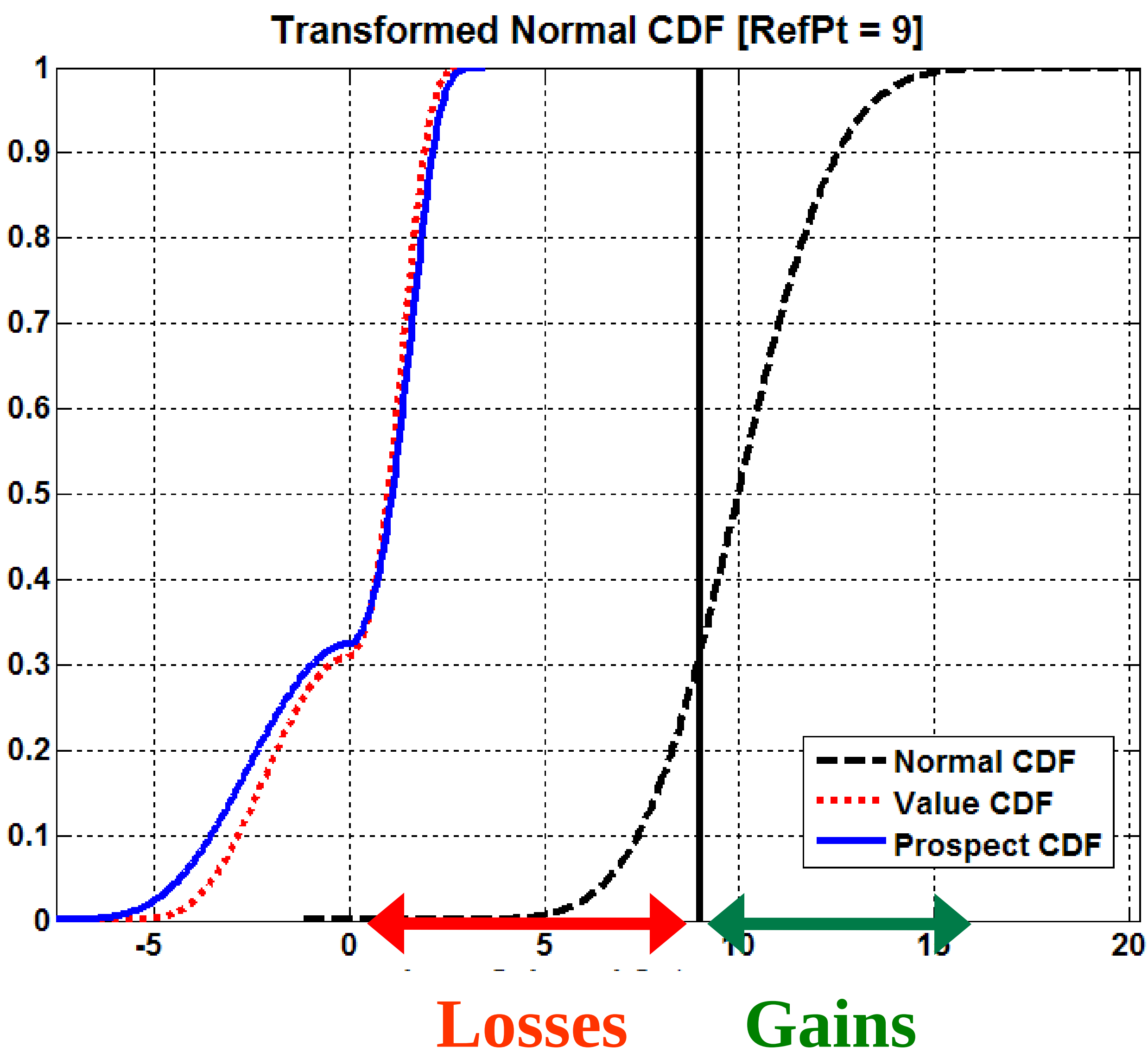
Value Function Maps Outcome Utilities



CPT Transformations



Utility: Cumulative Distribution


Transformed Normal CDF [RefPt = 9]
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-5
0
5
10
15
20
Normal CDF
Value CDF
Prospect CDF
Losses
Gains

Cumulative Prospect Value

CPV for continuous distribution of outcomes

$$cpv = \int_{u_0}^{\infty} -\frac{dw\left(1 - F_U\left(u\right)\right)}{du} \cdot g\left(u\right) du \;+ \int_{-\infty}^{u_0} \frac{dw\left(F_U\left(u\right)\right)}{du} \cdot g\left(u\right) du$$

$\boldsymbol{F}_U(u)$ = Path Utility CDF (the outcome distribution)
$g(.)$ = Value Function
$w(.)$ = Weight Function
u_0 = Reference Point

Two-link Network

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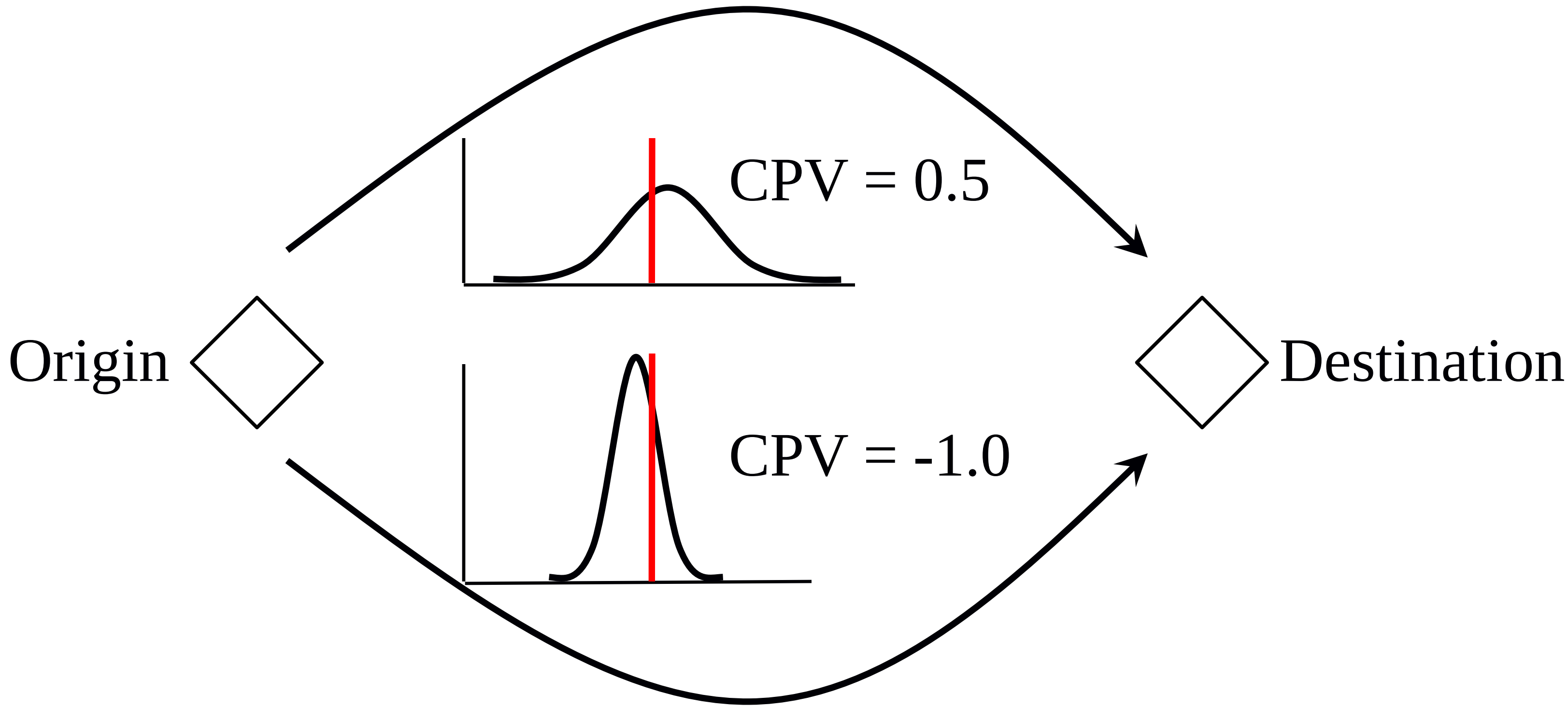


Choose between alternatives according to CPV

Making Variability Endogenous

- Part B Paper
  - Stochastic link travel times with exogenously defined distribution
  - Reference point (and other CPT parameters) assumed
  - Compute equilibrium: show dependence on RefPt and other parameters
- Extension here – to endogenize variability
  - **Stochastic OD demand**…
  - …split into path flows via ratio of means f/q hence stochastic path flows
  - Gives rise to stochastic link flows (with correlations)
  - Independently **stochastic link capacities**
  - Can compute resulting stochastic travel times (with correlations)
  - CPT applied to route choice

Stochastic Demand

Lognormal Demand $Q \sim LN(\mu,\sigma)$ with mean q

Split Q into path flows $F_k = (f_k/q)\cdot Q$

Where f_k is mean flow on path k

Path Flows Lognormal Distributed $F_k \sim LN(\mu_k,\sigma_k)$

Conservation condition on means: $q = \sum_k f_k$

Gives stochastic, correlated link flows.

Stochastic Supply

Link capacities are assumed to be independent random variables

$$\boxed{T_1 = t_a^0 + b_a \left(\frac{X_a}{C_a} \right)^{n_a}}$$

With $C_a \sim LN(\mu_{ca},\sigma_{ca})$ and independent of X_a

We therefore have non-trivial covariance matrices for link flows and link travel times.

Path costs arise from standard link-additive model.

BUT CPV is not link-additive!

Equilibrium Condition

With $cpv_k(x)$ the flow dependent CPV on route k we have that (demand feasible) **f*** is a CPV-UE if and only if:

$$\mathbf{cpv}\left(\mathbf{f}^*\right)^T\left(\mathbf{f}-\mathbf{f}^*\right) \le 0 \quad \forall \mathbf{f} \in F$$

$$\min_{\mathbf{f}\in\Omega}\left\{G\left(\mathbf{f}\right)\right\}$$

$$G\left(\mathbf{f}\right) = \max_{\mathbf{g}\in\Omega}\left(\mathbf{cpv}\left(\mathbf{f}\right)^T \cdot \mathbf{g} - \mathbf{cpv}\left(\mathbf{f}\right)^T \cdot \mathbf{f}\right)$$

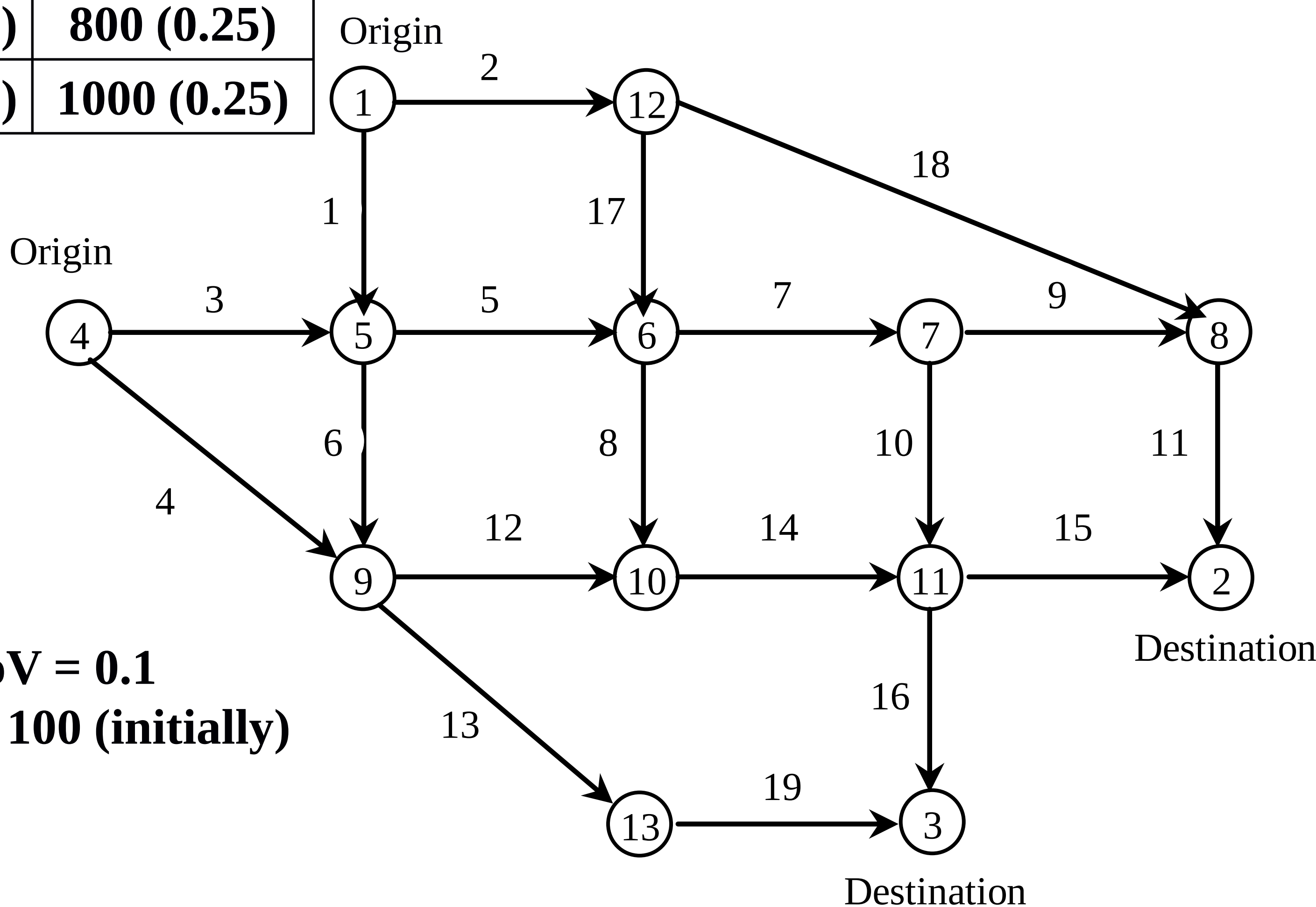
Equilibrium assignment algorithm

- Generate path set. Find initial feasible mean path flow **f**
- Compute travel time distribution for each path.
- Compute prospect value for each path & find ‘best’ path
- Assign mean OD flow to best path; gives auxiliary flow **g**
- Step from **f** toward **g** (as in MSA)
- Test convergence

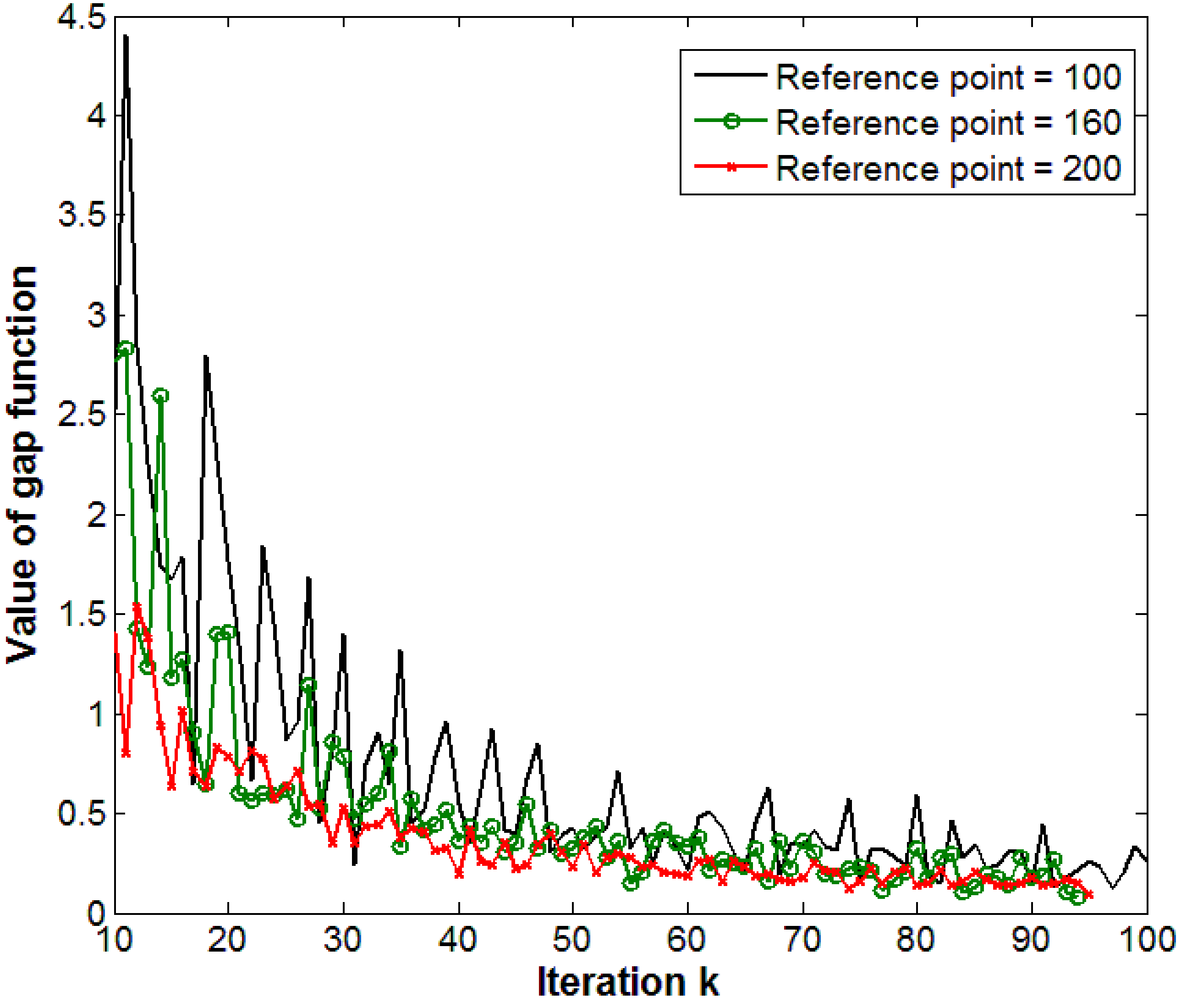
Nguyen and Dupuis Network

OD demand (CoV)

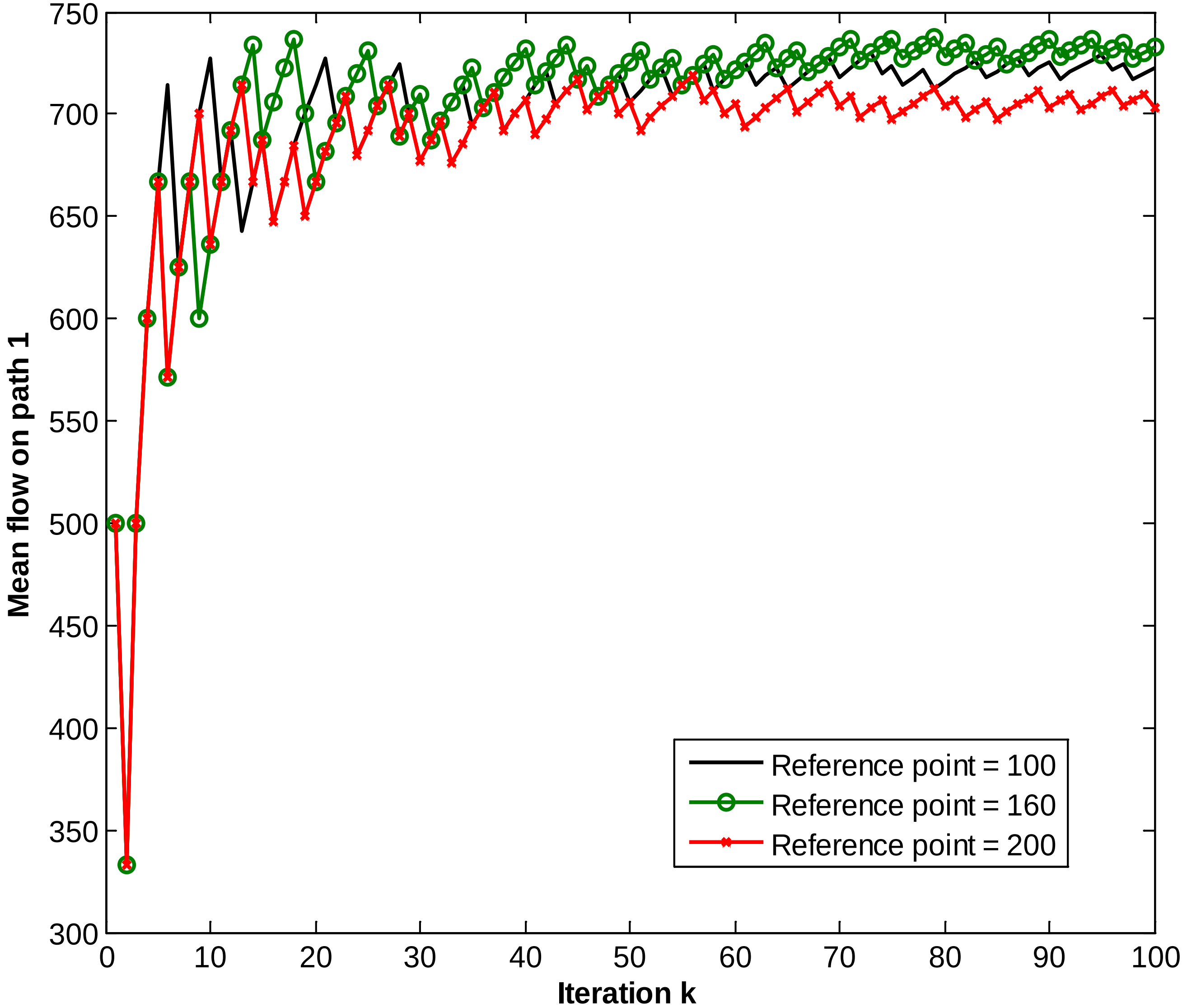
Orig¥Dest	2	3
1	1000 (0.2)	800 (0.25)
4	1500 (0.2)	1000 (0.25)



Link Capacities CoV = 0.1
Reference Points = 100 (initially)

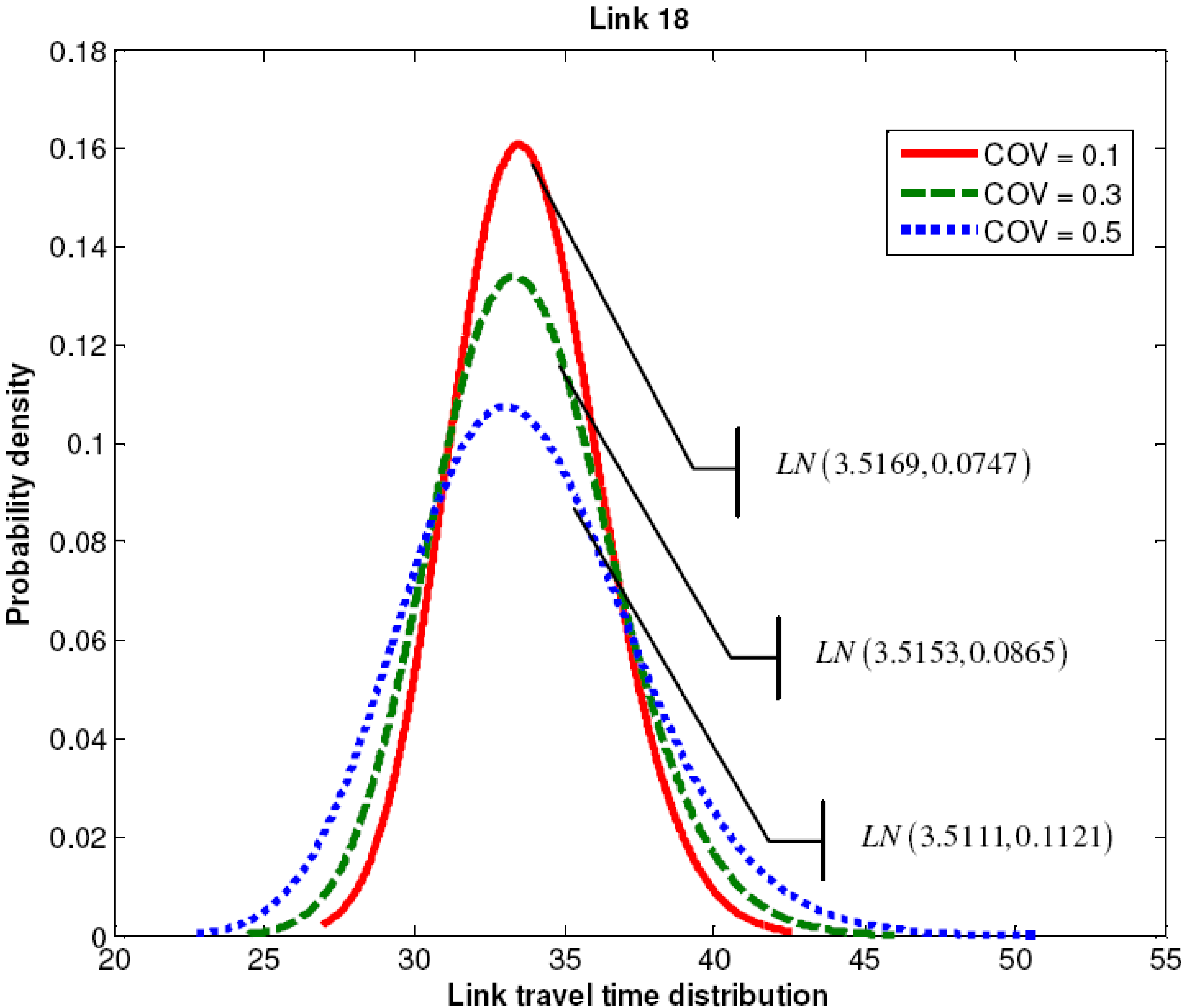


Reference point = 100
Reference point = 160
Reference point = 200
Value of gap function
Iteration k
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
10
20
30
40
50
60
70
80
90
100

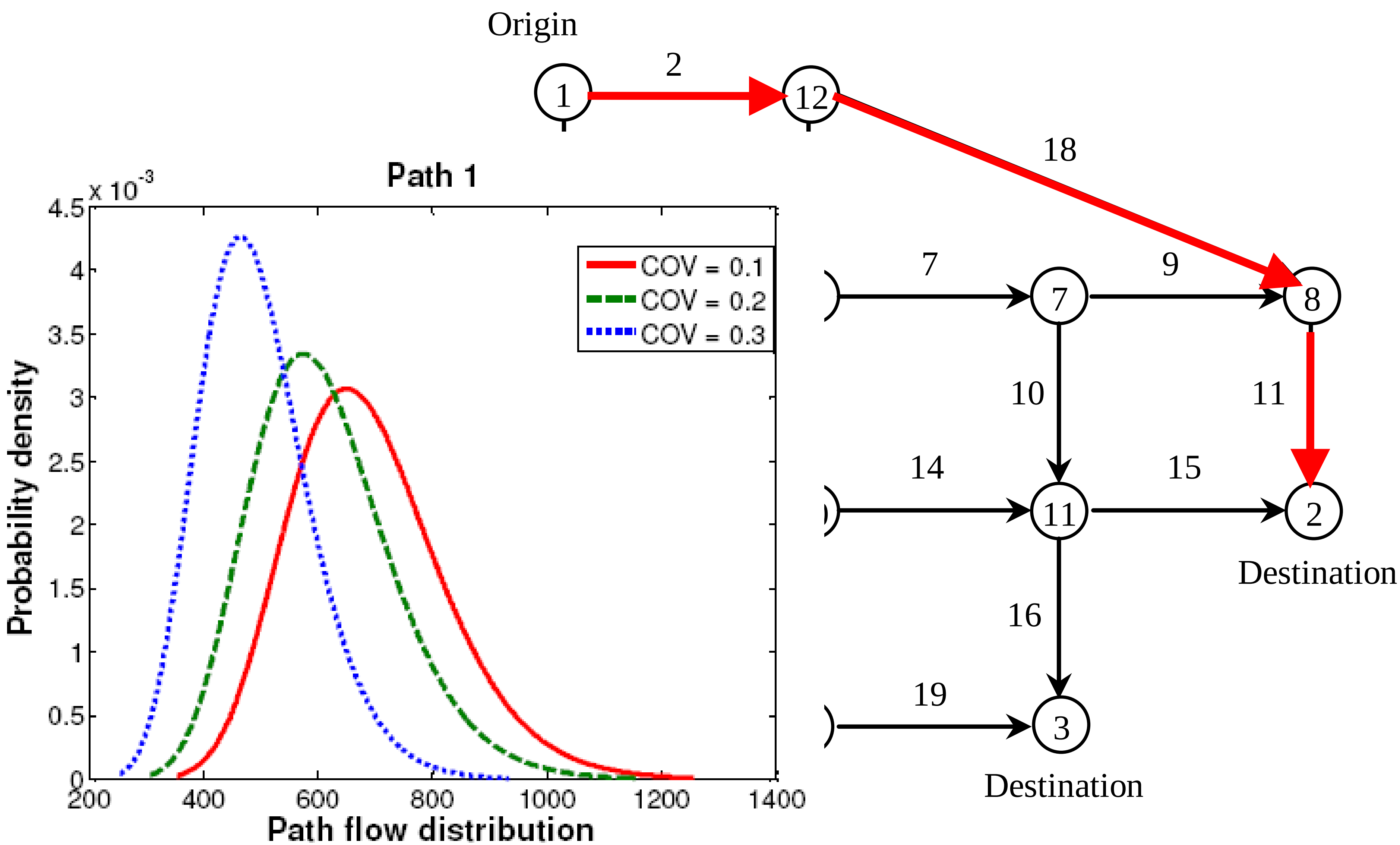


750
700
650
600
550
500
450
400
350
300
Mean flow on path 1
0
10
20
30
40
50
60
70
80
90
100
Iteration k
Reference point = 100
Reference point = 160
Reference point = 200

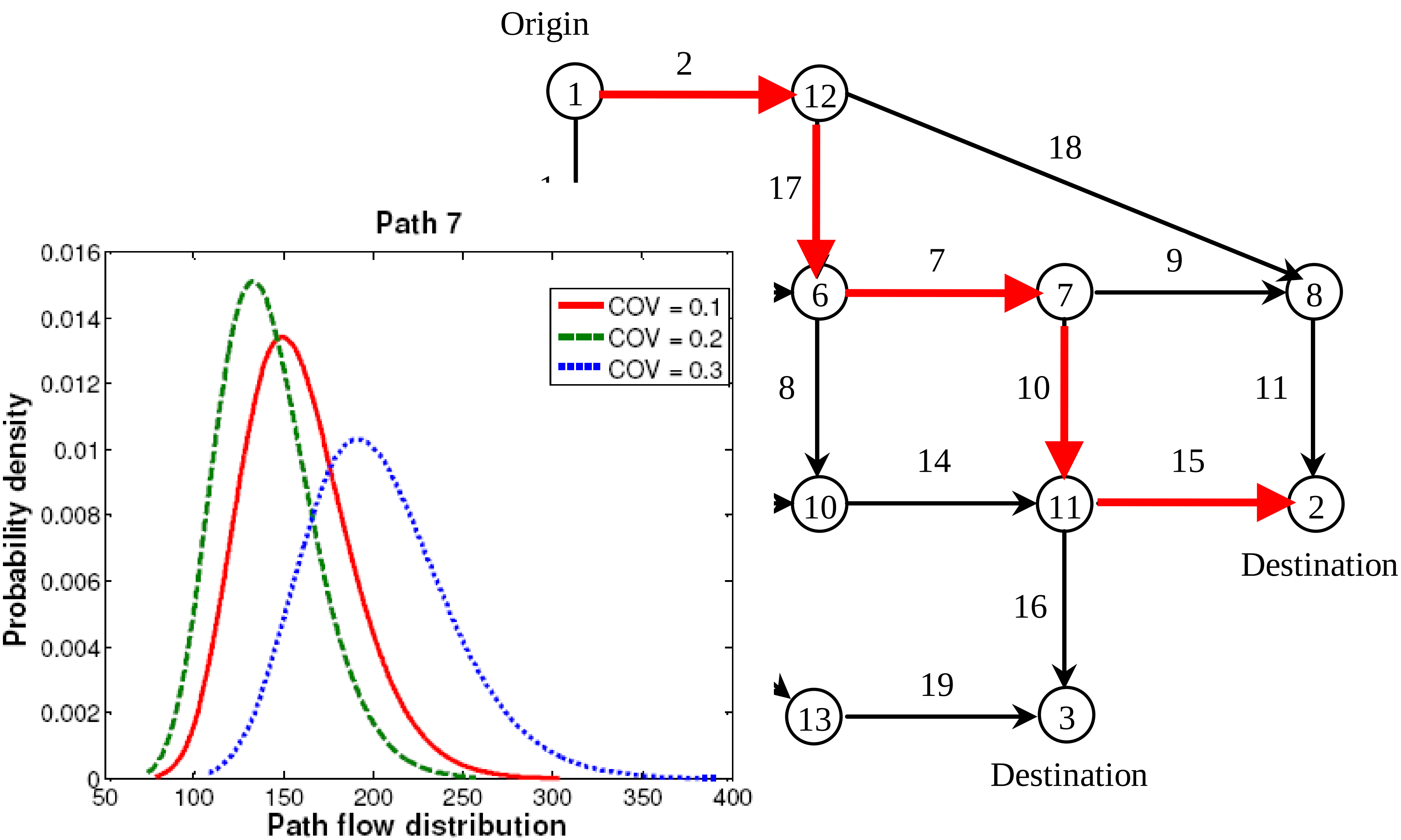
OD movement	OD pair	Path	Link sequence	PV	Flow		Travel time	
					Expected	SD	Expected	SD
1	(1,2)	1	2-18-11	49.188	685	137	61.578	5.090
		2	1-5-7-9-11	49.144	150	30	63.680	4.784
		3	1-5-7-10-15	-	-	-	-	-
		4	1-5-8-14-15	-	-	-	-	-
		5	1-6-12-14-15	-	-	-	-	-
		6	2-17-7-9-11	-	-	-	-	-
		7	2-17-7-10-15	49.123	165	33	64.000	4.729
		8	2-17-8-14-15	-	-	-	-	-
2	(4,2)	9	4-12-14-15	40.976	310	62	67.106	6.154
		10	3-5-7-9-11	40.967	411	82	68.130	6.086
		11	3-5-7-10-15	40.928	294	59	68.621	6.057
		12	3-5-8-14-15	40.970	169	34	68.398	6.067
		13	3-6-12-14-15	40.965	316	63	68.675	6.048
3	(1,3)	14	1-6-13-19	23.643	351	88	60.576	4.516
		15	1-5-7-10-16	23.626	10	0	61.846	4.062
		16	1-5-8-14-16	23.660	95	24	61.623	4.136
		17	1-6-12-14-16	-	-	-	-	-
		18	2-17-7-10-16	23.649	243	61	61.675	4.121
		19	2-17-8-14-16	23.659	101	25	61.452	4.206



Link 18
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
20
25
30
35
40
45
50
55
Probability density
Link travel time distribution
COV = 0.1
COV = 0.3
COV = 0.5
LN(3.5169,0.0747)
LN(3.5153,0.0865)
LN(3.5111,0.1121)



Origin
Destination
Destination
Path 1
x 10^-3
Probability density
Path flow distribution
COV = 0.1
COV = 0.2
COV = 0.3



Origin
Destination
Destination
2
18
17
7
9
8
10
11
14
15
16
19
Path 7
COV = 0.1
COV = 0.2
COV = 0.3
Probability density
Path flow distribution

Summary

Cumulative Prospect Theory as paradigm for route choice with stochastic travel times.

Continuous formulation of cumulative prospect theory presented.

Equilibrium condition defined.

Potential future studies:

- Solve for equilibrium efficiently and consider larger networks
    - **Path based** criterion so needs thought
- Network inference (estimation) from the observed link/path data
- Investigate impact of assuming CPV over E[Umax] or ‘reliability’ terms
    - How is UE recovered from CPV in the limit?
- Define reference point via a day-to-day learning process?
    - Initial reference point updated by experience, new alternatives added to choice set as reference point changes?
    - Does this converge? To a sensible solution? Is this process stable?