NETWORK EQUILIBRIUM UNDER CUMULATIVE PROSPECT THEORY WITH ENDOGENOUS STOCHASTIC DEMAND & SUPPLY

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Based on the paper Sumalee et al (2009) presented at ISTTT
How do drivers choose between routes with uncertain travel times?
Approaches

Mean Travel Time ⇒ Route 1

Prob Late Arrival ⇒ Route 2

Mean + stdDev
Mean + Variance
95th Percentile of TT distribution

…
Empirical Evidence leads to PT

• Choice modellers have shown that *decisions under uncertainty* do not conform to maximising the expected benefit.

• The empirical evidence suggests two major modifications to ‘Expected Utility Maximisation’:
  
  • the carriers of value are gains/losses relative to a reference point
  
  • the value of each outcome is multiplied by a decision weight, not by an additive probability.

  This generalisation of EUmax is Prospect Theory
Context

• A traffic network has many sources of variability in demand and supply: results in Travel Time Variability.

• Drivers are aware that travel times are uncertain and include this TTV in their decision making.


Empirical evidence

In this study the quasi-gain region. Estimation of the parameters of the value functions has also indicated that departure time decision is consistent with prospect theory.

Prospect Theory Transformations

The fundamental embodiment (and impact) of PT is through the following two transformations:

Value Function $g(.)$  

Probability Weighting Function $w(.)$

![Graph showing Value Function and Probability Weighting Function with Reference Point](image)
Empirical studies show that people’s preferences over final outcomes depend on the reference point from which they are judged:

- Kahneman & Tversky (1979) for decision under risk
- Kahneman *et al* (1990) for choice among commodity bundles
- Loewenstein & Prelec (1992) for inter-temporal choice
- Bateman *et al* (1997) for contingent valuation
- Dolan & Robinson (2001) for welfare theory
- Bleichrodt & Pinto (2002) for multi-attribute utility
- Senbil & Kitamura (2004) for departure time choice
- Jou, Kitamura *et al* (2008) for departure time choice
If we have a distribution of path costs $C_k \sim \mathcal{N}(c_k(x), \sigma_k)$, it is convenient to consider the utility

$$U_k = U_{\text{dest}} - C_k = U_{\text{dest}} - c_k(x) - \varepsilon_k$$
Applying the CPT Transforms

Weighting Function Maps Probabilities

Value Function Maps Outcome Utilities
CPT Transformations

Utility: Cumulative Distribution
Cumulative Prospect Value

CPV for continuous distribution of outcomes

\[ cpv = \int_{u_0}^{\infty} -\frac{dw(1 - F_U(u))}{du} \cdot g(u) \, du + \int_{-\infty}^{u_0} \frac{dw(F_U(u))}{du} \cdot g(u) \, du \]

\( F_U(u) \) = Path Utility CDF (the outcome distribution)
\( g(.) \) = Value Function
\( w(.) \) = Weight Function
\( u_0 \) = Reference Point
Two-link Network

How do drivers choose between routes with uncertain travel times?

Choose between alternatives according to CPV

CPV = 0.5
CPV = -1.0
Making Variability Endogenous

• Part B Paper
  ▪ Stochastic link travel times with exogenously defined distribution
  ▪ Reference point (and other CPT parameters) assumed
  ▪ Compute equilibrium: show dependence on RefPt and other parameters

• Extension here – to endogenize variability
  ▪ Stochastic OD demand…
  ▪ …split into path flows via ratio of means $f/q$ hence stochastic path flows
  ▪ Gives rise to stochastic link flows (with correlations)
  ▪ Independently stochastic link capacities
  ▪ Can compute resulting stochastic travel times (with correlations)
  ▪ CPT applied to route choice
Stochastic Demand

Lognormal Demand $Q \sim LN(\mu, \sigma)$ with mean $q$

Split $Q$ into path flows $F_k = (f_k/q) \cdot Q$

Where $f_k$ is mean flow on path $k$

Path Flows Lognormal Distributed $F_k \sim LN(\mu_k, \sigma_k)$

Conservation condition on means: $q = \sum_k f_k$

Gives stochastic, correlated link flows.
Stochastic Supply

Link capacities are assumed to be independent random variables

\[
T_1 = t_a^0 + b_a \left( \frac{X_a}{C_a} \right)^{n_a}
\]

With \( C_a \sim LN(\mu_{ca}, \sigma_{ca}) \) and independent of \( X_a \)

We therefore have non-trivial covariance matrices for link flows and link travel times.

Path costs arise from standard link-additive model.

BUT CPV is not link-additive!
With $cpv_k(x)$ the flow dependent CPV on route $k$ we have that (demand feasible) $f^*$ is a CPV-UE if and only if:

$$cpv(f^*)^T(f - f^*) \leq 0 \quad \forall f \in F$$

$$\min_{f \in \Omega} \{ G(f) \}$$

$$G(f) = \max_{g \in \Omega} \left( cpv(f)^T \cdot g - cpv(f)^T \cdot f \right)$$
Equilibrium assignment algorithm

• Generate path set. Find initial feasible mean path flow $f$
• Compute travel time distribution for each path.
• Compute prospect value for each path & find ‘best’ path
• Assign mean OD flow to best path; gives auxiliary flow $g$
• Step from $f$ toward $g$ (as in MSA)
• Test convergence
Nguyen and Dupuis Network

OD demand (CoV)

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Link Capacities CoV = 0.1
Reference Points = 100 (initially)
Value of gap function

Iteration k

Reference point = 100
Reference point = 160
Reference point = 200
Iteration $k$

Mean flow on path 1

Reference point = 100
Reference point = 160
Reference point = 200
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Summary

Cumulative Prospect Theory as paradigm for route choice with stochastic travel times.

Continuous formulation of cumulative prospect theory presented.

Equilibrium condition defined.

Potential future studies:

- Solve for equilibrium efficiently and consider larger networks
  - Path based criterion so needs thought
- Network inference (estimation) from the observed link/path data
- Investigate impact of assuming CPV over \( E[U_{\text{max}}] \) or ‘reliability’ terms
  - How is UE recovered from CPV in the limit?
- Define reference point via a day-to-day learning process?
  - Initial reference point updated by experience, new alternatives added to choice set as reference point changes?
  - Does this converge? To a sensible solution? Is this process stable?